3. $y = \frac{1}{a}\int_0^x f(t)\sin a(x-t)\,dt = \frac{1}{a}\int_0^x f(t)\sin ax\cos at\,dt - \frac{1}{a}\int_0^x f(t)\cos ax\sin at\,dt$

$= \frac{\sin ax}{a}\int_0^x f(t)\cos at\,dt - \frac{\cos ax}{a}\int_0^x f(t)\sin at\,dt \Rightarrow \frac{dy}{dx}$

$= \cos ax\left(\int_0^x f(t)\cos at\,dt\right) + \frac{\sin ax}{a}\left(\frac{d}{dx}\int_0^x f(t)\cos at\,dt\right) + \sin ax\int_0^x f(t)\sin at\,dt - \frac{\cos ax}{a}\left(\frac{d}{dx}\int_0^x f(t)\sin at\,dt\right)$

$= \cos ax\int_0^x f(t)\cos at\,dt + \frac{\sin ax}{a}(f(x)\cos ax) + \sin ax\int_0^x f(t)\sin at\,dt - \frac{\cos ax}{a}(f(x)\sin ax)$

$\Rightarrow \frac{dy}{dx} = \cos ax\int_0^x f(t)\cos at\,dt + \sin ax\int_0^x f(t)\sin at\,dt$. Next, $\frac{d^2y}{dx^2} =$

$-a\sin ax\int_0^x f(t)\cos at\,dt + (\cos ax)\left(\frac{d}{dx}\int_0^x f(t)\cos at\,dt\right) + a\cos ax\int_0^x f(t)\sin at\,dt + (\sin ax)\left(\frac{d}{dx}\int_0^x f(t)\sin at\,dt\right)$

$= -a\sin ax\int_0^x f(t)\cos at\,dt + (\cos ax)f(x)\cos ax + a\cos ax\int_0^x f(t)\sin at\,dt + (\sin ax)f(x)\sin ax$

$= -a\sin ax\int_0^x f(t)\cos at\,dt + a\cos ax\int_0^x f(t)\sin at\,dt + f(x)$. Therefore, $y'' + a^2y$

$= a\cos ax\int_0^x f(t)\sin at\,dt - a\sin ax\int_0^x f(t)\cos at\,dt + f(x) + a^2\left(\frac{\sin ax}{a}\int_0^x f(t)\cos at\,dt - \frac{\cos ax}{a}\int_0^x f(t)\sin at\,dt\right)$

$= f(x)$. Note also that $y'(0) = y(0) = 0$.

4. $x = \int_0^y \frac{1}{\sqrt{1+4t^2}}\,dt \Rightarrow \frac{d}{dx}(x) = \frac{d}{dx}\int_0^y \frac{1}{\sqrt{1+4t^2}}\,dt = \frac{d}{dy}\left[\int_0^y \frac{1}{\sqrt{1+4t^2}}\,dt\right]\left(\frac{dy}{dx}\right)$ from the chain rule

$\Rightarrow 1 = \frac{1}{\sqrt{1+4y^2}}\left(\frac{dy}{dx}\right) \Rightarrow \frac{dy}{dx} = \sqrt{1+4y^2}$. Then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\sqrt{1+4y^2}\right) = \frac{d}{dy}\left(\sqrt{1+4y^2}\right)\left(\frac{dy}{dx}\right)$

$= \frac{1}{2}\left(1+4y^2\right)^{-1/2}(8y)\left(\frac{dy}{dx}\right) = \frac{4y\left(\frac{dy}{dx}\right)}{\sqrt{1+4y^2}} = \frac{4y\left(\sqrt{1+4y^2}\right)}{\sqrt{1+4y^2}} = 4y$. Thus $\frac{d^2y}{dx^2} = 4y$, and the constant of proportionality is 4.

5. (a) $\int_0^{x^2} f(t)\,dt = x\cos\pi x \Rightarrow \frac{d}{dx}\int_0^{x^2} f(t)\,dt = \cos\pi x - \pi x\sin\pi x \Rightarrow f(x^2)(2x) = \cos\pi x - \pi x\sin\pi x$

$\Rightarrow f(x^2) = \frac{\cos\pi x - \pi x\sin\pi x}{2x}$. Thus, $x = 2 \Rightarrow f(4) = \frac{\cos 2\pi - 2\pi\sin 2\pi}{4} = \frac{1}{4}$

(b) $\int_0^{f(x)} t^2\,dt = \left[\frac{t^3}{3}\right]_0^{f(x)} = \frac{1}{3}\left(f(x)\right)^3 \Rightarrow \frac{1}{3}\left(f(x)\right)^3 = x\cos\pi x \Rightarrow \left(f(x)\right)^3 = 3x\cos\pi x \Rightarrow f(x) = \sqrt[3]{3x\cos\pi x}$

$\Rightarrow f(4) = \sqrt[3]{3(4)\cos 4\pi} = \sqrt[3]{12}$

6. $\int_0^a f(x)\,dx = \frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a$. Let $F(a) = \int_0^a f(t)\,dt \Rightarrow f(a) = F'(a)$. Now $F(a) = \frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a$

$\Rightarrow f(a) = F'(a) = a + \frac{1}{2}\sin a + \frac{a}{2}\cos a - \frac{\pi}{2}\sin a \Rightarrow f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \frac{1}{2}\sin\frac{\pi}{2} + \frac{\left(\frac{\pi}{2}\right)}{2}\cos\frac{\pi}{2} - \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2} + \frac{1}{2} - \frac{\pi}{2} = \frac{1}{2}$

7. $\int_1^b f(x)\,dx = \sqrt{b^2+1} - \sqrt{2} \Rightarrow f(b) = \frac{d}{db}\int_1^b f(x)\,dx = \frac{1}{2}(b^2+1)^{-1/2}(2b) = \frac{b}{\sqrt{b^2+1}} \Rightarrow f(x) = \frac{x}{\sqrt{x^2+1}}$

8. The derivative of the left side of the equation is: $\frac{d}{dx}\left[\int_0^x\left[\int_0^u f(t)\,dt\right]du\right]=\int_0^x f(t)\,dt$; the derivative of the right side of the equation is: $\frac{d}{dx}\left[\int_0^x f(u)(x-u)\,du\right]=\frac{d}{dx}\int_0^x f(u)\,x\,du-\frac{d}{dx}\int_0^x u\,f(u)\,du$

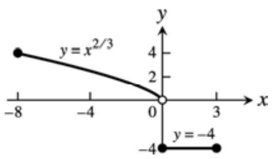
$=\frac{d}{dx}\left[x\int_0^x f(u)\,du\right]-\frac{d}{dx}\int_0^x u\,f(u)\,du=\int_0^x f(u)\,du+x\left[\frac{d}{dx}\int_0^x f(u)\,du\right]-x f(x)=\int_0^x f(u)\,du+x f(x)-x f(x)$

$=\int_0^x f(u)\,du$. Since each side has the same derivative, they differ by a constant, and since both sides equal 0 when $x=0$, the constant must be 0. Therefore, $\int_0^x\left[\int_0^u f(t)\,dt\right]du=\int_0^x f(u)(x-u)\,du$.

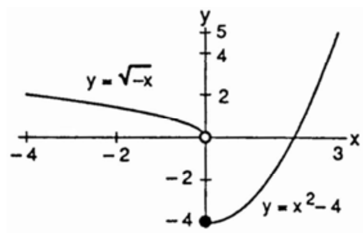
9. $\frac{dy}{dx}=3x^2+2\Rightarrow y=\int(3x^2+2)\,dx=x^3+2x+C$. Then (1, –1) lies on the curve $\Rightarrow 1^3+2(1)+C=-1$
$\Rightarrow C=-4\Rightarrow y=x^3+2x-4$

10. The acceleration due to gravity downward is –32 ft/sec^2 $\Rightarrow v=\int -32\,dt=-32t+v_0$, where v_0 is the initial velocity $\Rightarrow v=-32t+32\Rightarrow s=\int(-32t+32)\,dt=-16t^2+32t+C$. If the release point, at $t=0$, is $s=0$, then $C=0\Rightarrow s=-16t^2+32t$. Then $s=17\Rightarrow 17=-16t^2+32t\Rightarrow 16t^2-32t+17=0$. The discriminant of this quadratic equation is –64 which says there is no real time when $s=17$ ft. You had better duck.

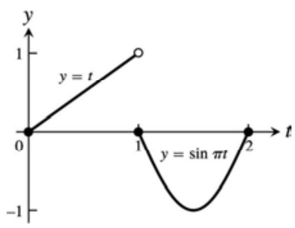
11. $\int_{-8}^{3} f(x)\,dx=\int_{-8}^{0} x^{2/3}dx+\int_0^3 -4\,dx$
$=\left[\frac{3}{5}x^{5/3}\right]_{-8}^{0}+[-4x]_0^3$
$=\left(0-\frac{3}{5}(-8)^{5/3}\right)+(-4(3)-0)=\frac{96}{5}-12=\frac{36}{5}$



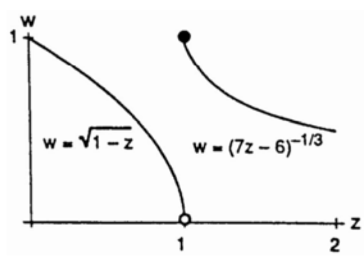
12. $\int_{-4}^{3} f(x)\,dx=\int_{-4}^{0}\sqrt{-x}\,dx+\int_0^3(x^2-4)\,dx$
$=\left[-\frac{2}{3}(-x)^{3/2}\right]_{-4}^{0}+\left[\frac{x^3}{3}-4x\right]_0^3$
$=\left[0-\left(-\frac{2}{3}(4)^{3/2}\right)\right]+\left[\left(\frac{3^3}{3}-4(3)\right)-0\right]=\frac{16}{3}-3=\frac{7}{3}$



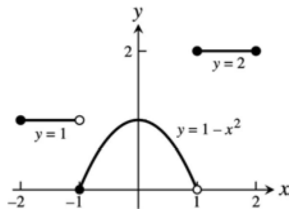
13. $\int_0^2 g(t)\,dt=\int_0^1 t\,dt+\int_1^2 \sin \pi t\,dt$
$=\left[\frac{t^2}{2}\right]_0^1+\left[-\frac{1}{\pi}\cos \pi t\right]_1^2$
$=\left(\frac{1}{2}-0\right)+\left[-\frac{1}{\pi}\cos 2\pi-\left(-\frac{1}{\pi}\cos \pi\right)\right]=\frac{1}{2}-\frac{2}{\pi}$



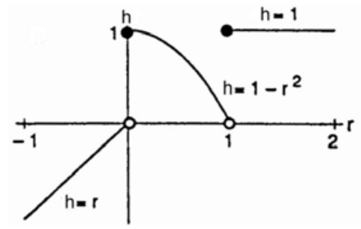
14. $\int_0^2 h(z)\,dz=\int_0^1\sqrt{1-z}\,dz+\int_1^2(7z-6)^{-1/3}\,dz$
$=\left[-\frac{2}{3}(1-z)^{3/2}\right]_0^1+\left[\frac{3}{14}(7z-6)^{2/3}\right]_1^2$
$=\left[-\frac{2}{3}(1-1)^{3/2}-\left(-\frac{2}{3}(1-0)^{3/2}\right)\right]$
$+\left[\frac{3}{14}(7(2)-6)^{2/3}-\frac{3}{14}(7(1)-6)^{2/3}\right]$
$=\frac{2}{3}+\left(\frac{6}{7}-\frac{3}{14}\right)=\frac{55}{42}$



15. $\int_{-2}^{2} f(x)\,dx = \int_{-2}^{-1} dx + \int_{-1}^{1} (1-x^2)\,dx + \int_{1}^{2} 2\,dx$
$= [x]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^{1} + [2x]_1^2$
$= (-1-(-2)) + \left[\left(1-\frac{1^3}{3}\right) - \left(-1-\frac{(-1)^3}{3}\right)\right] + [2(2)-2(1)]$
$= 1 + \frac{2}{3} - \left(-\frac{2}{3}\right) + 4 - 2 = \frac{13}{3}$



16. $\int_{-1}^{2} h(r)\,dr = \int_{-1}^{0} r\,dr + \int_{0}^{1} (1-r^2)\,dr + \int_{1}^{2} dr$
$= \left[\frac{r^2}{2}\right]_{-1}^{0} + \left[r - \frac{r^3}{3}\right]_{0}^{1} + [r]_1^2$
$= \left(0 - \frac{(-1)^2}{2}\right) + \left(\left(1-\frac{1^3}{3}\right) - 0\right) + (2-1) = -\frac{1}{2} + \frac{2}{3} + 1 = \frac{7}{6}$



17. Ave. value $= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-0}\int_0^2 f(x)\,dx = \frac{1}{2}\left[\int_0^1 x\,dx + \int_1^2 (x-1)\,dx\right] = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^2$
$= \frac{1}{2}\left[\left(\frac{1^2}{2} - 0\right) + \left(\frac{2^2}{2} - 2\right) - \left(\frac{1^2}{2} - 1\right)\right] = \frac{1}{2}$

18. Ave. value $= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{3-0}\int_0^3 f(x)\,dx = \frac{1}{3}\left[\int_0^1 dx + \int_1^2 0\,dx + \int_2^3 dx\right] = \frac{1}{3}[1-0+0+3-2] = \frac{2}{3}$

19. $\lim\limits_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}}\,dx = \lim\limits_{b\to 1^-} [\sin^{-1} x]_0^b = \lim\limits_{b\to 1^-} (\sin^{-1} b - \sin^{-1} 0) = \lim\limits_{b\to 1^-} (\sin^{-1} b - 0) = \lim\limits_{b\to 1^-} \sin^{-1} b = \frac{\pi}{2}$

20. $\lim\limits_{x\to\infty} \frac{1}{x}\int_0^x \tan^{-1} t\,dt = \lim\limits_{x\to\infty} \frac{\int_0^x \tan^{-1} t\,dt}{x}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$
$= \lim\limits_{x\to\infty} \frac{\tan^{-1} x}{1} = \frac{\pi}{2}$

21. $\lim\limits_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) = \lim\limits_{n\to\infty}\left(\left(\frac{1}{n}\right)\left[\frac{1}{1+\left(\frac{1}{n}\right)}\right] + \left(\frac{1}{n}\right)\left[\frac{1}{1+2\left(\frac{1}{n}\right)}\right] + \cdots + \left(\frac{1}{n}\right)\left[\frac{1}{1+n\left(\frac{1}{n}\right)}\right]\right)$ which can be interpreted as a Riemann sum with partitioning $\Delta x = \frac{1}{n} \Rightarrow \lim\limits_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) = \int_0^1 \frac{1}{1+x}\,dx = [\ln(1+x)]_0^1 = \ln 2$

22. $\lim\limits_{n\to\infty} \frac{1}{n}[e^{1/n} + e^{2/n} + \cdots + e] = \lim\limits_{n\to\infty}\left[\left(\frac{1}{n}\right)e^{(1/n)} + \left(\frac{1}{n}\right)e^{2(1/n)} + \cdots + \left(\frac{1}{n}\right)e^{n(1/n)}\right]$ which can be interpreted as a Riemann sum with partitioning $\Delta x = \frac{1}{n} \Rightarrow \lim\limits_{n\to\infty} \frac{1}{n}[e^{1/n} + e^{2/n} + \cdots + e] = \int_0^1 e^x\,dx = [e^x]_0^1 = e - 1$

23. Let $f(x) = x^5$ on $[0, 1]$. Partition $[0, 1]$ into n subintervals with $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. Then $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$ are the right-hand endpoints of the subintervals. Since f is increasing on $[0, 1]$, $U = \sum\limits_{j=1}^{\infty}\left(\frac{j}{n}\right)^5\left(\frac{1}{n}\right)$ is the upper sum for $f(x) = x^5$ on $[0, 1] \Rightarrow \lim\limits_{n\to\infty}\sum\limits_{j=1}^{\infty}\left(\frac{j}{n}\right)^5\left(\frac{1}{n}\right) = \lim\limits_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \ldots + \left(\frac{n}{n}\right)^5\right] = \lim\limits_{n\to\infty}\left[\frac{1^5+2^5+\ldots+n^5}{n^6}\right] = \int_0^1 x^5\,dx = \left[\frac{x^6}{6}\right]_0^1 = \frac{1}{6}$

24. Let $f(x)=x^3$ on [0, 1]. Partition [0, 1] into n subintervals with $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. Then $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$ are the right-hand endpoints of the subintervals. Since f is increasing on [0, 1], $U = \sum_{j=1}^{\infty}\left(\frac{j}{n}\right)^3\left(\frac{1}{n}\right)$ is the upper sum for $f(x)=x^3$ on $[0, 1] \Rightarrow \lim_{n\to\infty}\sum_{j=1}^{\infty}\left(\frac{j}{n}\right)^3\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^3+\left(\frac{2}{n}\right)^3+\ldots+\left(\frac{n}{n}\right)^3\right] = \lim_{n\to\infty}\left[\frac{1^3+2^3+\ldots+n^3}{n^4}\right] = \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

25. Let $y=f(x)$ on [0, 1]. Partition [0, 1] into n subintervals with $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. Then $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$ are the right-hand endpoints of the subintervals. Since f is continuous on [0, 1], $\sum_{j=1}^{\infty} f\left(\frac{j}{n}\right)\left(\frac{1}{n}\right)$ is a Riemann sum of $y=f(x)$ on $[0, 1] \Rightarrow \lim_{n\to\infty}\sum_{j=1}^{\infty} f\left(\frac{j}{n}\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f\left(\frac{n}{n}\right)\right] = \int_0^1 f(x)\,dx$

26. (a) $\lim_{n\to\infty}\frac{1}{n^2}[2+4+6+\ldots+2n] = \lim_{n\to\infty}\frac{1}{n}\left[\frac{2}{n}+\frac{4}{n}+\frac{6}{n}+\ldots+\frac{2n}{n}\right] = \int_0^1 2x\,dx = [x^2]_0^1 = 1$, where $f(x)=2x$ on [0, 1] (see Exercise 21)

(b) $\lim_{n\to\infty}\frac{1}{n^{16}}[1^{15}+2^{15}+\ldots+n^{15}] = \lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^{15}+\left(\frac{2}{n}\right)^{15}+\ldots+\left(\frac{n}{n}\right)^{15}\right] = \int_0^1 x^{15}\,dx = \left[\frac{x^{16}}{16}\right]_0^1 = \frac{1}{16}$, where $f(x)=x^{15}$ on [0, 1] (see Exercise 21)

(c) $\lim_{n\to\infty}\frac{1}{n}\left[\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\ldots+\sin\frac{n\pi}{n}\right] = \int_0^1 \sin n\pi\,dx = \left[-\frac{1}{\pi}\cos\pi x\right]_0^1 = -\frac{1}{\pi}\cos\pi - \left(-\frac{1}{\pi}\cos 0\right) = \frac{2}{\pi}$, where $f(x)=\sin\pi x$ on [0, 1] (see Exercise 21)

(d) $\lim_{n\to\infty}\frac{1}{n^{17}}\left[1^{15}+2^{15}+\ldots+n^{15}\right] = \left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\frac{1}{n^{16}}[1^{15}+2^{15}+\ldots+n^{15}]\right) = \left(\lim_{n\to\infty}\frac{1}{n}\right)\int_0^1 x^{15}dx = 0\left(\frac{1}{16}\right) = 0$ (see part (b) above)

(e) $\lim_{n\to\infty}\frac{1}{n^{15}}\left[1^{15}+2^{15}+\ldots+n^{15}\right] = \lim_{n\to\infty}\frac{n}{n^{16}}[1^{15}+2^{15}+\ldots+n^{15}]$
$= \left(\lim_{n\to\infty} n\right)\left(\lim_{n\to\infty}\frac{1}{n^{16}}[1^{15}+2^{15}+\ldots+n^{15}]\right) = \left(\lim_{n\to\infty} n\right)\int_0^1 x^{15}dx = \infty$ (see part (b) above)

27. (a) Let the polygon be inscribed in a circle of radius r. If we draw a radius from the center of the circle (and the polygon) to each vertex of the polygon, we have n isosceles triangles formed (the equal sides are equal to r, the radius of the circle) and a vertex angle of θ_n where $\theta_n = \frac{2\pi}{n}$. The area of each triangle is $A_n = \frac{1}{2}r^2\sin\theta_n \Rightarrow$ the area of the polygon is $A = nA_n = \frac{nr^2}{2}\sin\theta_n = \frac{nr^2}{2}\sin\frac{2\pi}{n}$.

(b) $\lim_{n\to\infty} A = \lim_{n\to\infty}\frac{nr^2}{2}\sin\frac{2\pi}{n} = \lim_{n\to\infty}\frac{n\pi r^2}{2\pi}\sin\frac{2\pi}{n} = \lim_{n\to\infty}\left(\pi r^2\right)\frac{\sin\left(\frac{2\pi}{n}\right)}{\left(\frac{2\pi}{n}\right)} = \left(\pi r^2\right)\lim_{2\pi/n\to\infty}\frac{\sin\left(\frac{2\pi}{n}\right)}{\left(\frac{2\pi}{n}\right)} = \pi r^2$

28. Partition [0, 1] into n subintervals, each of length $\Delta x = \frac{1}{n}$ with the points $x_0 = 0, x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \ldots, x_n = \frac{n}{n} = 1$. The inscribed rectangles so determined have areas $f(x_0)\,\Delta x = (0)^2\,\Delta x$, $f(x_1)\Delta x = \left(\frac{1}{n}\right)^2\Delta x$, $f(x_2)\,\Delta x = \left(\frac{2}{n}\right)^2\Delta x, \ldots, f(x_{n-1}) = \left(\frac{n-1}{n}\right)^2\Delta x$. The sum of these areas is
$S_n = \left(0^2+\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\ldots+\left(\frac{n-1}{n}\right)^2\right)\Delta x = \left(\frac{1^2}{n^2}+\frac{2^2}{n^2}+\ldots+\frac{(n-1)^2}{n^2}\right)\frac{1}{n} = \frac{1^2}{n^3}+\frac{2^2}{n^3}+\ldots+\frac{(n-1)^2}{n^3}$. Then
$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1^2}{n^3}+\frac{2^2}{n^3}+\ldots+\frac{(n-1)^2}{n^3}\right) = \int_0^1 x^2dx = \frac{1^3}{3} = \frac{1}{3}$.

29. (a) $g(1)=\int_1^1 f(t)\,dt=0$

(b) $g(3)=\int_1^3 f(t)\,dt=-\frac{1}{2}(2)(1)=-1$

(c) $g(-1)=\int_1^{-1} f(t)\,dt=-\int_{-1}^1 f(t)\,dt=-\frac{1}{4}(\pi 2^2)=-\pi$

(d) $g'(x)=f(x)=0\Rightarrow x=-3,1,3$ and the sign chart for $g'(x)=f(x)$ is $\underset{-3}{|}+++\underset{1}{|}---\underset{3}{|}+++$. So g has a relative maximum at $x=1$.

(e) $g'(-1)=f(-1)=2$ is the slope and $g(-1)=\int_1^{-1} f(t)dt=-\pi$, by (c). Thus the equation is $y+\pi=2(x+1)$ $\Rightarrow y=2x+2-\pi$.

(f) $g''(x)=f'(x)=0$ at $x=-1$ and $g''(x)=f'(x)$ is negative on $(-3,-1)$ and positive on $(-1,1)$ so there is an inflection point for g at $x=-1$. We notice that $g''(x)=f'(x)<0$ for x on $(-1,2)$ and $g''(x)=f'(x)>0$ for x on $(2,4)$, even though $g''(2)$ does not exist, g has a tangent line at $x=2$, so there is an inflection point at $x=2$.

(g) g is continuous on $[-3,4]$ and so it attains its absolute maximum and minimum values on this interval. We saw in (d) that $g'(x)=0\Rightarrow x=-3,1,3$. We have that $g(-3)=\int_1^{-3} f(t)\,dt=-\int_{-3}^1 f(t)\,dt=-\frac{\pi 2^2}{2}=-2\pi$

$g(1)=\int_1^1 f(t)\,dt=0$

$g(3)=\int_1^3 f(t)\,dt=-1$

$g(4)=\int_1^4 f(t)\,dt=-1+\frac{1}{2}\cdot 1\cdot 1=-\frac{1}{2}$

Thus, the absolute minimum is -2π and the absolute maximum is 0. Thus, the range is $[-2\pi, 0]$.

30. $y=\sin x+\int_x^\pi \cos 2t\,dt+1=\sin x-\int_\pi^x \cos 2t\,dt+1\Rightarrow y'=\cos x-\cos(2x)$; when $x=\pi$ we have $y'=\cos\pi-\cos(2\pi)=-1-1=-2$. And $y''=-\sin x+2\sin(2x)$; when $x=\pi$, $y=\sin\pi+\int_x^\pi \cos 2t\,dt+1=0+0+1=1$.

31. $f(x)=\int_{1/x}^x \frac{1}{t}\,dt\Rightarrow f'(x)=\frac{1}{x}\left(\frac{dx}{dx}\right)-\left(\frac{1}{\frac{1}{x}}\right)\left(\frac{d}{dx}\left(\frac{1}{x}\right)\right)=\frac{1}{x}-x\left(-\frac{1}{x^2}\right)=\frac{1}{x}+\frac{1}{x}=\frac{2}{x}$

32. $f(x)=\int_{\cos x}^{\sin x}\frac{1}{1-t^2}\,dt\Rightarrow f'(x)=\left(\frac{1}{1-\sin^2 x}\right)\left(\frac{d}{dx}(\sin x)\right)-\left(\frac{1}{1-\cos^2 x}\right)\left(\frac{d}{dx}(\cos x)\right)=\frac{\cos x}{\cos^2 x}+\frac{\sin x}{\sin^2 x}=\frac{1}{\cos x}+\frac{1}{\sin x}$

33. $g(y)=\int_{\sqrt{y}}^{2\sqrt{y}}\sin t^2\,dt\Rightarrow g'(y)=\left(\sin\left(2\sqrt{y}\right)^2\right)\left(\frac{d}{dy}\left(2\sqrt{y}\right)\right)-\left(\sin\left(\sqrt{y}\right)^2\right)\left(\frac{d}{dy}\left(\sqrt{y}\right)\right)=\frac{\sin 4y}{\sqrt{y}}-\frac{\sin y}{2\sqrt{y}}$

34. $g(y)=\int_{\sqrt{y}}^{y^2}\frac{e^t}{t}\,dt\Rightarrow g'(y)=\frac{e^{y^2}}{y^2}\cdot\frac{d}{dy}(y^2)-\frac{e^{\sqrt{y}}}{\sqrt{y}}\cdot\frac{d}{dy}\left(\sqrt{y}\right)=\frac{e^{y^2}}{y^2}(2y)-\frac{e^{\sqrt{y}}}{\sqrt{y}}\cdot\frac{1}{2\sqrt{y}}=\frac{4e^{y^2}}{2y}-\frac{e^{\sqrt{y}}}{2y}=\frac{4e^{y^2}-e^{\sqrt{y}}}{2y}$

35. $y=\int_{x^2/2}^{x^2}\ln\sqrt{t}\,dt\Rightarrow\frac{dy}{dx}=\left(\ln\sqrt{x^2}\right)\cdot\frac{d}{dx}(x^2)-\left(\ln\sqrt{\frac{x^2}{2}}\right)\cdot\frac{d}{dx}\left(\frac{x^2}{2}\right)=2x\ln|x|-x\ln\frac{|x|}{\sqrt{2}}$

36. $y=\int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t\,dt$

$\Rightarrow \frac{dy}{dx}=\left(\ln\sqrt[3]{x}\right)\cdot\frac{d}{dx}\left(\sqrt[3]{x}\right)-\left(\ln\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)=\left(\ln\sqrt[3]{x}\right)\left(\frac{1}{3}x^{-2/3}\right)-\left(\ln\sqrt{x}\right)\left(\frac{1}{2}x^{-1/2}\right)=\frac{\ln\sqrt[3]{x}}{3\sqrt[3]{x^2}}-\frac{\ln\sqrt{x}}{2\sqrt{x}}\sqrt[3]{x}$

37. $\int_0^{\ln x}\sin e^t\,dt \Rightarrow y'=(\sin e^{\ln x})\cdot\frac{d}{dx}(\ln x)=\frac{\sin x}{x}$

38. $y=\int_{e^{4\sqrt{x}}}^{e^{2x}}\ln t\,dt \Rightarrow y'=(\ln e^{2x})\cdot\frac{d}{dx}(e^{2x})-\left(\ln e^{4\sqrt{x}}\right)\cdot\frac{d}{dx}\left(e^{4\sqrt{x}}\right)=(2x)(2e^{2x})-\left(4\sqrt{x}\right)\left(e^{4\sqrt{x}}\right)\cdot\frac{d}{dx}\left(4\sqrt{x}\right)$

$=4xe^{2x}-4\sqrt{x}e^{4\sqrt{x}}\left(\frac{2}{\sqrt{x}}\right)=4xe^{2x}-8e^{4\sqrt{x}}$

39. $f(x)=\int_x^{x+3}t(5-t)\,dt \Rightarrow f'(x)=(x+3)(5-(x+3))\left(\frac{d}{dx}(x+3)\right)-x(5-x)\left(\frac{dx}{dx}\right)=(x+3)(2-x)-x(5-x)$
$=6-x-x^2-5x+x^2=6-6x$. Thus $f'(x)=0\Rightarrow 6-6x=0\Rightarrow x=1$. Also, $f''(x)=-6<0\Rightarrow x=1$ gives a maximum.

40. $\ln x^{(x^x)}=x^x\ln x$ and $\ln(x^x)^x=x\ln x^x=x^2\ln x$; then $x^x\ln x=x^2\ln x\Rightarrow(x^x-x^2)\ln x=0\Rightarrow x^x=x^2$ or $\ln x=0$. $\ln x=0\Rightarrow x=1$; $x^x=x^2\Rightarrow x\ln x=2\ln x\Rightarrow x=2$. Therefore, $x^{(x^x)}=(x^x)^x$ when $x=2$ or $x=1$.

41. $A_1=\int_1^e\frac{2\log_2 x}{x}dx=\frac{2}{\ln 2}\int_1^e\frac{\ln x}{x}dx=\left[\frac{(\ln x)^2}{\ln 2}\right]_1^e=\frac{1}{\ln 2}$;

$A_2=\int_1^e\frac{2\log_4 x}{4}dx=\frac{2}{\ln 4}\int_1^e\frac{\ln x}{x}dx=\left[\frac{(\ln x)^2}{2\ln 2}\right]_1^e=\frac{1}{2\ln 2}\Rightarrow A_1:A_2=2:1$

42. (a) $\frac{df}{dx}=\frac{2\ln e^x}{e^x}\cdot e^x=2x$

(b) $f(0)=\int_1^1\frac{2\ln t}{t}dt=0$

(c) $\frac{df}{dx}=2x\Rightarrow f(x)=x^2+C$; $f(0)=0\Rightarrow C=0\Rightarrow f(x)=x^2\Rightarrow$ the graph of $f(x)$ is a parabola.

43. $f(x)=e^{g(x)}\Rightarrow f'(x)=e^{g(x)}g'(x)$, where $g'(x)=\frac{x}{1+x^4}\Rightarrow f'(2)=e^0\left(\frac{2}{1+16}\right)=\frac{2}{17}$

44. The area of the blue shaded region is $\int_0^1\sin^{-1}x\,dx=\int_0^1\sin^{-1}y\,dy$, which is the same as the area of the region to the left of the curve $y=\sin x$ (and part of the rectangle formed by the coordinate axes and dashed lines $y=1$, $x=\frac{\pi}{2}$). The area of the rectangle is $\frac{\pi}{2}=\int_0^1\sin^{-1}y\,dy+\int_0^{\pi/2}\sin x\,dx$, so we have

$\frac{\pi}{2}=\int_0^1\sin^{-1}x\,dx+\int_0^{\pi/2}\sin x\,dx\Rightarrow\int_0^{\pi/2}\sin x\,dx=\frac{\pi}{2}-\int_0^1\sin^{-1}x\,dx.$

45. (a) slope of L_3 < slope of L_2 < slope of $L_1\Rightarrow\frac{1}{b}<\frac{\ln b-\ln a}{b-a}<\frac{1}{a}$

(b) area of small (shaded) rectangle < area under curve < area of large rectangle

$\Rightarrow\frac{1}{b}(b-a)<\int_a^b\frac{1}{x}dx<\frac{1}{a}(b-a)\Rightarrow\frac{1}{b}<\frac{\ln b-\ln a}{b-a}<\frac{1}{a}$

46. (a) If f is continuously differentiable on $[a, b]$, then so is the function $g(x) = (x-c)f(x)$. So the two integrals on the right side exist and

$$\int_a^c (x-c)f(x)\,dx + \int_c^b (x-c)f(x)\,dx = \int_a^b (x-c)f(x)\,dx$$
$$= \int_a^b x\,f(x)\,dx - \int_a^b c\,f(x)\,dx = \int_a^b x\,f(x)\,dx - c(0) = \int_a^b x\,f(x)\,dx$$

(b) Split the right side in part (a) into two integrals and write it as $-\int_0^{\ell} t\,f(c-t)\,dt + \int_0^{\ell} t\,f(c+t)\,dt.$ For the first integral above, use the substitution $t = c - x$, $\Rightarrow x = c - t$, $dt = -dx$; when $t = 0$, $x = c$ and when $t = \ell = (b-a)/2$, $x = (a+b)/2 - (b-a)/2 = a$. (Note that when x is in $[a, c]$, $c - x$ is positive and thus $c - x$ agrees in sign with t.) $-\int_0^{\ell} t\,f(c-t)\,dt = \int_{\ell}^{0} t\,f(c-t)\,dt = \int_a^c -(x-c)f(x)(-dx) = \int_a^c (x-c)f(x)\,dx.$

Thus the first integral above is equal to $\int_a^c (x-c)f(x)\,dx.$ For the second integral above, use the substitution $t = x - c, \Rightarrow c + t = x$ and $dt = dx$; when $t = 0, x = c$ and when $t = \ell = (b-a)/2$, $x = (b-a)/2 + (a+b)/2 = b.$ Thus the second integral above is equal to $\int_c^b (x-c)f(x)\,dx$ and

$$\int_a^b xf(x)\,dx = \int_a^c (x-c)f(x)\,dx + \int_c^b (x-c)f(x)\,dx = \int_0^{\ell} t\,f(c+t)\,dt - \int_0^{\ell} t\,f(c-t)\,dt$$
$$= \int_0^{\ell} t(f(c+t) - f(c-t))\,dt.$$

(c) According to the mean value theorem of Section 4.2, for every t in $(0, \ell)$, there is a point in q in $(c-t, c+t)$ at which $\dfrac{f(c+t)-f(c-t)}{(c+t)-(c-t)} = \dfrac{f(c+t)-f(c-t)}{2t} = f'(q).$ Since for all these q belonging to each t, we have $f'(q) \le M$, $f(c+t) - f(c-t) \le 2tM$ for all t in $(0, \ell)$. Thus

$$\left|\int_a^b x\,f(x)\,dx\right| = \left|\int_0^{\ell} t(f(c+t) - f(c-t))\,dt\right| \le \int_0^{\ell} (t)(2tM) = \frac{2}{3}M\,t^3\Big]_0^{(b-a)/2} = \frac{(b-a)^3}{12}M.$$