حل $u=\sqrt{x}\Rightarrow x=u^2\Rightarrow dx=2udu\Rightarrow$

$$\int\frac{dx}{x-\sqrt{x}}=\int\frac{2udu}{u^2-u}=2\int\frac{du}{u-1}=2Ln\mid u-1\mid+C$$

$$=2Ln\mid\sqrt{x}-1\mid+C\Rightarrow\int_4^9\frac{dx}{x-\sqrt{x}}=2Ln\mid\sqrt{x}-1\mid\Big]_4^9$$
$$=2(Ln2-Ln1)=2Ln2$$

۱۱. $\int\frac{dx}{xLnx},\ u=Lnx$

حل $u=Lnx\Rightarrow du=\frac{1}{x}dx\Rightarrow\int\frac{dx}{xLnx}=\int\frac{du}{u}$

$$=Ln|u|+C=Ln|Lnx|+C$$

۱۲. $\int_0^2\frac{e^x dx}{1+e^x},\ u=1+e^x$

حل $u=1+e^x\Rightarrow du=e^x dx\Rightarrow\int\frac{e^x dx}{1+e^x}$

$$=\int\frac{du}{u}=Ln\mid u\mid+C=Ln\mid 1+e^x\mid+C$$
$$\Rightarrow\int_0^2\frac{e^x dx}{1+e^x}=Ln\mid 1+e^x\mid\Big]_0^2$$
$$=Ln(1+e^2)-Ln2=Ln\left(\frac{1+e^2}{2}\right)$$

۱۳. $\int_0^1 e^{\sqrt{x}}dx,\ u=\sqrt{x}$

حل $u=\sqrt{x}\Rightarrow u^2=x\Rightarrow 2udu=dx\Rightarrow$

$$\int e^{\sqrt{x}}dx=\int 2ue^u du=2(ue^u-e^u)+C$$
$$=2\left(\sqrt{x}e^{\sqrt{x}}-e^{\sqrt{x}}\right)+C=2e^{\sqrt{x}}(\sqrt{x}-1)+C$$
$$\Rightarrow I=2e^{\sqrt{x}}(\sqrt{x}-1)\Big]_0^1=2$$

۱۴. $\int\frac{\tan\sqrt{x}}{\sqrt{x}}dx,\quad u=\sqrt{x}$

حل $u=\sqrt{x}\Rightarrow du=\frac{dx}{2\sqrt{x}}\Rightarrow\frac{dx}{\sqrt{x}}=2du\Rightarrow$

$$\int\frac{\tan\sqrt{x}}{\sqrt{x}}dx=2\int\tan u du=2Ln\mid secu\mid+C$$
$$=2Ln\mid\sec\sqrt{x}\mid+c$$

۱۵. $\int\tan x\sec^2 x dx,\ u=\tan x$

حل $\Rightarrow du=\sec^2 x dx\Rightarrow\int\tan x\sec^2 x dx=\int u du$

$$=\frac{1}{2}u^2+C=\frac{1}{2}\tan^2 x+C$$

۱۶. $\int\tan x\sec^2 x dx,\ u=\sec x$

حل $\Rightarrow du=\sec x\tan x dx\Rightarrow\int\tan x\sec^2 x dx$

$$=\int u du=\frac{1}{2}u^2+C=\frac{1}{2}\sec^2 x+C$$

در مسائل ۱۷-۲۰، انتگرالها را به کمک اتحادها و جانشانیهای داده شده به صورت متداول درآورید و سپس آنها را محاسبه کنید.

۱۷. $\int Cos^2 x dx,\ Cos^2 x=1-Sin^2 x,\ u=Sinx$

حل $\Rightarrow du=Cosx dx\ \&\ \int Cos^2 x dx=\int Cos^2 x Cosx dx$

$$=\int(1-Sin^2 x)Cosx dx=\int(1-u^2)du$$
$$=u-\frac{1}{3}u^3+C=Sinx-\frac{Sin^3 x}{3}+C$$

۱۸. $\int Sin^2 x Cos^2 x dx,\ Cos^2 x=1-Sin^2 x,\ u=Sinx$

حل $\Rightarrow du=Cosx dx\Rightarrow\int Sin^2 x Cos^2 x dx$

$$=\int Sin^2 x(1-Sin^2 x)Cosx dx=\int u^2(1-u^2)du$$
$$=\frac{1}{5}u^5-\frac{1}{7}u^7+C=\frac{1}{5}Sin^5 x-\frac{Sin^7 x}{7}+C$$

۱۹. $\int\tan^2 x\sec x dx,\ \tan^2 x=\sec^2 x-1\ \ u=\sec x$

حل $\Rightarrow du=\sec x\tan x dx\Rightarrow\int\tan^2 x\sec x dx$

$$=\int\tan^2 x\tan x\sec x dx=\int(\sec^2 x-1)\tan x\sec x dx$$
$$=\int(u^2-1)du=\frac{u^3}{3}-u+C=\frac{1}{3}\sec^3 x-\sec x+C$$

۲۰. $\int\tan^2 x\sec^2 x dx,\ \tan^2 x=\sec^2 x-1\quad u=\sec x$

حل $\Rightarrow du=\sec x\tan x dx\Rightarrow\int\tan^2 x\sec^2 x dx$

$$=\int(\sec^2 x-1)\tan x\sec^2 x\sec x dx=\int(u^2-1)u^2 du$$
$$=\frac{u^5}{5}-\frac{u^3}{3}+C=\frac{\sec^5 x}{5}-\frac{\sec^3 x}{3}+C$$

در مسائل ۲۱-۸۰، جانشانیهایی به دست آورید که بتوان به کمک آنها انتگرالها را به صورتهای متداول درآورد. سپس انتگرال‌ها را محاسبه کنید.

۲۱. $\int\sqrt{2x+3}\,dx$

حل $u=2x+3\Rightarrow du=2dx\Rightarrow\int\sqrt{2x+3}\,dx=\frac{1}{2}\int\sqrt{u}\,du$

$$=\frac{۲}{۳}u^{\frac{۳}{۲}}+C=\frac{۲}{۳}(\text{Sin}x)^{\frac{۳}{۲}}+C$$

۲۸. $\int \frac{\sec^۲x}{۲+\tan x}dx$

حل $u=۲+\tan x\Rightarrow du=\sec^۲x dx\Rightarrow$

$$\int\frac{\sec^۲x\,dx}{۲+\tan x}=\int\frac{du}{u}=\text{Ln}\mid ۲+\tan x\mid +C$$

۲۹. $\int \frac{\text{Cos}x}{(۱+\text{Sin}x)^۲}dx$

حل $u=۱+\text{Sin}x\Rightarrow du=\text{Cos}x\,dx\Rightarrow \int\frac{\text{Cos}x\,dx}{(۱+\text{Sin}x)^۲}$

$$=\int\frac{du}{u^۲}=\frac{-۱}{u}+C=\frac{-۱}{۱+\text{Sin}x}+C$$

۳۰. $\int_{۰}^{\pi/۴} ۸\text{Cos}^۳۲x\text{Sin}۲x\,dx$

حل $u=\text{Cos}۲x\Rightarrow du=-۲\text{Sin}۲x dx\Rightarrow$

$$\int ۸\text{Cos}^۳۲x\text{Sin}۲x\,dx=\frac{-۱}{۲}\int ۸u^۳du=-u^۴+C=-\text{Cos}^۴۲x+C$$

$$\Rightarrow I=-\text{Cos}^۴۲x\Big]_{۰}^{\frac{\pi}{۴}}=۰-(-۱)=۱$$

۳۱. $\int_{-\pi/۲}^{\pi/۲} \text{Sin}^۲۲x\text{Sin}^۳۲x\,dx$

(راهنمایی: $\text{Cos}^۳۲x=\text{Cos}۲x(۱-\text{Sin}^۲۲x)$.)

حل $u=\text{Sin}۲x\Rightarrow du=۲\text{Cos}۲x dx\Rightarrow \int \text{Sin}^۲۲x\text{Cos}^۳۲x\,dx$

$$=\int \text{Sin}^۲۲x(۱-\text{Sin}^۲۲x)\text{Cos}۲x\,dx=\frac{۱}{۲}\int(u^۲-u^۴)du$$

$$=\frac{۱}{۶}u^۳-\frac{۱}{۱۰}u^۵+C=\frac{۱}{۶}\text{Sin}^۳۲x-\frac{۱}{۱۰}\text{Sin}^۵۲x+C$$

$$\Rightarrow I=\frac{۱}{۶}\text{Sin}^۳۲x-\frac{۱}{۱۰}\text{Sin}^۵۲x\Big]_{\frac{-\pi}{۲}}^{\frac{\pi}{۲}}=۰$$

نتیجه فوق از فرد بودن تابع هم واضح است چون بازه متقارن است.

۳۲. $\int \csc^۳x\,dx$ (راهنمایی $\csc^۳x=\csc^۲x(۱+\cot^۲x)$)

حل $=\int \csc^۳x\,dx=\int \csc^۲x(۱+\cot^۲x)dx$

$$=\int \csc^۲x\,dx+\int \csc^۲x\cot^۲x+C$$

$$=-\cot x-\frac{۱}{۳}\cot^۳x+C$$

۳۳. $\int \frac{x\,dx}{\sqrt{۱-۴x^۲}}$

حل $u=۱-۴x^۲\Rightarrow du=-۸x dx\Rightarrow \int\frac{x}{\sqrt{۱-۴x^۲}}dx$

$$=-\frac{۱}{۴}u^{\frac{۳}{۲}}+C=\frac{۱}{۴}\left(۲x+۳\right)^{\frac{۳}{۲}}+C$$

$$\Rightarrow I=\frac{۱}{۴}\left(۲x+۳\right)^{\frac{۳}{۲}}\Big]_{\frac{۱}{۲}}^{۲}=\frac{۱۹}{۳}$$

۲۲. $\int_{۱}^{۴۰}\frac{dx}{۳x+۵}$

حل $u=۳x+۵\Rightarrow du=۳dx\Rightarrow \int\frac{dx}{۳x+۵}=\frac{۱}{۳}\int\frac{du}{u}$

$$=\frac{۱}{۳}\text{Ln}\mid u\mid +C=\frac{۱}{۳}\text{Ln}\mid ۳x+۵\mid +C$$

$$\Rightarrow I=\frac{۱}{۳}\text{Ln}\mid ۳x+۵\mid \Big]_{۱}^{۴۰}=\frac{۱}{۳}\{\text{Ln}۱۲۵-\text{Ln}۸\}$$

$$=\frac{۱}{۳}\{۳\text{Ln}۵-۳\text{Ln}۲\}=\text{Ln}۵-\text{Ln}۲=\text{Ln}\frac{۵}{۲}$$

۲۳. $\int\frac{dx}{(۲x-۷)^۲}$

حل $u=۲x-۷\Rightarrow du=۲dx\Rightarrow \int\frac{dx}{(۲x-۷)^۲}=\frac{۱}{۲}\int\frac{du}{u^۲}$

$$=\frac{-۱}{۲u}+C=\frac{-۱}{۲(۲x-۷)}+C$$

۲۴. $\int\frac{(x+۱)dx}{x^۲+۲x+۳}$

حل $u=x^۲++۳\Rightarrow du=۲(x+۱)dx\Rightarrow$

$$\int\frac{(x+۱)dx}{x^۲+۲x+۳}=\frac{۱}{۲}\int\frac{du}{u}=\frac{۱}{۲}\text{Ln}\mid u\mid +C$$

$$=\frac{۱}{۲}\text{Ln}\mid x^۲+۲x+۳\mid +C$$

۲۵. $\int_{-\pi}^{۰}\frac{\text{Sin}x\,dx}{۲+\text{Cos}x}$

حل $u=۲+\text{Cos}x\Rightarrow \text{Sin}x dx=-du\Rightarrow \int\frac{\text{Sin}x}{۲+\text{Cos}x}dx$

$$=-\int\frac{du}{u}=-\text{Ln}\mid u\mid =-\text{Ln}\mid ۲+\text{Cos}x\mid +C$$

$$\Rightarrow I=-\text{Ln}\mid ۲+\text{Cos}x\mid \Big]_{-\pi}^{۰}=-\text{Ln}۳+\text{Ln}۱=-\text{Ln}۳$$

۲۶. $\int \tan^۳۲x\sec^۲۲x\,dx$

حل if $u=\tan ۲x$ then we have $du=۲\sec^۲۲x dx$

$$\Rightarrow\int\tan^۳۲x\sec^۲۲x\,dx=\frac{۱}{۲}\int u^۳du$$

$$=\frac{u^۴}{۸}+C=\frac{۱}{۸}\tan^۴۲x+C$$

۲۷. $\int \sqrt{\text{Sin}x}\,\text{Cos}x\,dx$

حل $u=\text{Sin}x\Rightarrow du=\text{Cos}x dx\Rightarrow \int\sqrt{\text{Sin}x}\,\text{Cos}x\,dx=\int\sqrt{u}\,du$

$$= -\frac{1}{8}\int \frac{du}{\sqrt{u}} = \frac{-\sqrt{u}}{4} + C = \frac{-\sqrt{1-4x^2}}{4} + C$$

۳۴. $\int x^{1/3}\sqrt{x^{4/3}-1}\,dx$

حل $u = x^{\frac{4}{3}} - 1 \Rightarrow du = \frac{4}{3}x^{\frac{1}{3}}dx \Rightarrow \frac{3}{4}du = x^{\frac{1}{3}}dx$

$$\Rightarrow \int x^{\frac{1}{3}}\sqrt{x^{\frac{4}{3}}-1}\,dx = \frac{3}{4}\int \sqrt{u}\,du$$

$$= \frac{1}{2}u^{\frac{3}{2}} + C = \frac{1}{2}(x^{\frac{4}{3}}-1)^{\frac{3}{2}} + C$$

۳۵. $\int_{0}^{\frac{\sqrt{2}}{2}} \frac{dx}{\sqrt{1-x^2}}$

حل $\int_{0}^{\frac{\sqrt{2}}{2}} \frac{dx}{\sqrt{1-x^2}} = \text{Sin}^{-1}x\Big]_{0}^{\frac{\sqrt{2}}{2}} = \frac{\pi}{4}$

۳۶. $\int_{0}^{\pi/2} \text{Sin}\,2x\,dx$

حل ۱ $\int_{0}^{\frac{\pi}{2}} \text{Sin}\,2x\,dx = \frac{-1}{2}\text{Cos}\,2x\Big]_{0}^{\frac{\pi}{2}} = \frac{-1}{2}[-1-1] = 1$

۳۷. $\int_{0}^{1} \frac{3x}{1+x^2}dx$

حل $\Rightarrow \frac{3}{2}\int_{0}^{1} \frac{2x}{1+x^2}dx = \frac{3}{2}\text{Ln}\,|1+x^2|\Big]_{0}^{1} = \frac{3}{2}\text{Ln}\,2$

۳۸. $\int_{0}^{\pi} x\text{Sin}(x^2)dx$

حل $\int_{0}^{\pi} x\text{Sin}x^2dx = -\frac{1}{2}\text{Cos}x^2\Big]_{0}^{\pi} = \frac{1}{2}\left[1-\text{Cos}\pi^2\right]$

۳۹. $\int_{0}^{\pi/4} \frac{\text{Sin}^2 2x}{1+\text{Cos}\,2x}dx$ (راهنمایی: $\text{Sin}^2 2x = 1-\text{Cos}^2 2x$)

حل $\int \frac{\text{Sin}^2 2x}{1+\text{Cos}\,2x}dx = \int \frac{1-\text{Cos}^2 2x}{1+\text{Cos}\,2x}dx = \int (1-\text{Cos}\,2x)dx$

$$= x - \frac{1}{2}\text{Sin}\,2x + C \Rightarrow$$

$$I = x - \frac{1}{2}\text{Sin}\,2x\Big]_{0}^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{1}{2} - 0 = \frac{\pi-2}{4}$$

۴۰. $\int_{\sqrt{2}}^{2} \frac{dy}{y\sqrt{y^2-1}}$

حل $\int_{\sqrt{2}}^{2} \frac{dy}{y\sqrt{y^2-1}} = \text{sec}^{-1}|y|\Big]_{\sqrt{2}}^{2} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

۴۱. $\int_{0}^{1/2} \frac{2dx}{\sqrt{1-4x^2}}$

حل $\int_{0}^{\frac{1}{2}} \frac{2dx}{\sqrt{1-4x^2}} = \int_{0}^{\frac{1}{2}} \frac{2dx}{\sqrt{1-(2x)^2}} = \text{Sin}^{-1}(2x)\Big]_{0}^{\frac{1}{2}} = \frac{\pi}{2}$

۴۲. $\int_{0}^{1/\sqrt{2}} \frac{2vdv}{\sqrt{1-v^4}}$

حل $\int_{0}^{\frac{1}{\sqrt{2}}} \frac{2vdv}{\sqrt{1-v^4}} = \int_{0}^{\frac{1}{\sqrt{2}}} \frac{d(v)^2}{\sqrt{1-(v^2)^2}} = \text{Sin}^{-1}v^2\Big]_{0}^{\frac{1}{\sqrt{2}}} = \frac{\pi}{6}$

۴۳. $\int \frac{xdx}{(3x^2+4)^3}$

حل $u = 3x^2+4 \Rightarrow du = 6xdx \Rightarrow xdx = \frac{1}{6}du \Rightarrow$

$$\int \frac{xdx}{(3x^2+4)^3} = \frac{1}{6}\int \frac{du}{u^3} = \frac{-1}{12u^2} + c = \frac{-1}{12(3x^2+4)^2} + c$$

۴۴. $\int x^2\sqrt{x^3+5}\,dx$

حل $u = x^3+5 \Rightarrow du = 3x^2dx \Rightarrow x^2dx = \frac{1}{3}du$

$$\Rightarrow \int x^2\sqrt{x^3+5}\,dx = \frac{1}{3}\int \sqrt{u}\,du$$

$$= \frac{2}{9}u^{\frac{3}{2}} + c = \frac{2}{9}\left(x^3+5\right)^{\frac{3}{2}} + c$$

۴۵. $\int \frac{x^2dx}{\sqrt{x^3+5}}$

حل با توجه به تمرین قبل داریم:

$$\int \frac{x^2dx}{\sqrt{x^3+5}} = \frac{1}{3}\int \frac{du}{\sqrt{u}} = \frac{2}{3}\sqrt{u} + c = \frac{2}{3}\sqrt{x^3+5} + c$$

۴۶. $\int \frac{xdx}{4x^2+1}$

حل $u = 4x^2+1 \Rightarrow du = 8xdx \Rightarrow xdx = \frac{1}{8}du \Rightarrow$

$$\int \frac{xdx}{4x^2+1} = \frac{1}{8}\int \frac{du}{u} = \frac{1}{8}\text{Ln}\,|u| + c$$

$$= \frac{1}{8}\text{Ln}\,|4x^2+1| + c$$

۴۷. $\int_{0}^{\text{Ln}\,2} e^{2x}dx$

حل $\int_{0}^{\text{Ln}\,2} e^{2x}dx = \frac{1}{2}\int_{0}^{\text{Ln}\,2} 2e^{2x}dx = \frac{1}{2}e^{2x}\Big]_{0}^{\text{Ln}\,2}$

$$= \frac{1}{2}\left[e^{2\text{Ln}\,2} - e^0\right] = \frac{1}{2}[4-1] = \frac{3}{2}$$

۴۸. $\int e^{\text{Cos}x}\text{Sin}x\,dx$

حل $\int e^{\text{Cos}x}\text{Sin}x\,dx = -\int e^{\text{Cos}x}d(\text{Cos}x) = -e^{\text{Cos}x} + c$

۴۹. $\int \frac{dx}{e^{2x}}$