§ 15. Dual bases

One more word before embarking on the proofs of the important theorems. The concept of dual space was defined without any reference to coordinate systems; a glance at the following proofs will show a superabundance of coordinate systems. We wish to point out that this phenomenon is inevitable; we shall be establishing results concerning dimension, and dimension is the one concept (so far) whose very definition is given in terms of a basis.

THEOREM 1. *If* $\mathcal{V}$ *is an n-dimensional vector space, if* $\{x_1, \cdots, x_n\}$ *is a basis in* $\mathcal{V}$, *and if* $\{\alpha_1, \cdots, \alpha_n\}$ *is any set of n scalars, then there is one and only one linear functional y on* $\mathcal{V}$ *such that* $[x_i, y] = \alpha_i$ *for* $i = 1, \cdots, n$.

PROOF. Every x in $\mathcal{V}$ may be written in the form $x = \xi_1 x_1 + \cdots + \xi_n x_n$ in one and only one way; if y is any linear functional, then

$$[x, y] = \xi_1[x_1, y] + \cdots + \xi_n[x_n, y].$$

From this relation the uniqueness of y is clear; if $[x_i, y] = \alpha_i$, then the value of $[x, y]$ is determined, for every x, by $[x, y] = \sum_i \xi_i\alpha_i$. The argument can also be turned around; if we define y by

$$[x, y] = \xi_1\alpha_1 + \cdots + \xi_n\alpha_n,$$

then y is indeed a linear functional, and $[x_i, y] = \alpha_i$.

THEOREM 2. *If* $\mathcal{V}$ *is an n-dimensional vector space and if* $\mathcal{X} = \{x_1, \cdots, x_n\}$ *is a basis in* $\mathcal{V}$, *then there is a uniquely determined basis* $\mathcal{X}'$ *in* $\mathcal{V}'$, $\mathcal{X}' = \{y_1, \cdots, y_n\}$, *with the property that* $[x_i, y_j] = \delta_{ij}$. *Consequently the dual space of an n-dimensional space is n-dimensional.*

The basis $\mathcal{X}'$ is called the *dual basis* of $\mathcal{X}$.

PROOF. It follows from Theorem 1 that, for each $j = 1, \cdots, n$, a unique y_j in $\mathcal{V}'$ can be found so that $[x_i, y_j] = \delta_{ij}$; we have only to prove that the set $\mathcal{X}' = \{y_1, \cdots, y_n\}$ is a basis in $\mathcal{V}'$.

In the first place, $\mathcal{X}'$ is a linearly independent set, for if we had $\alpha_1 y_1 + \cdots + \alpha_n y_n = 0$, in other words, if

$$[x, \alpha_1 y_1 + \cdots + \alpha_n y_n] = \alpha_1[x, y_1] + \cdots + \alpha_n[x, y_n] = 0$$

for all x, then we should have, for $x = x_i$,

$$0 = \sum_j \alpha_j[x_i, y_j] = \sum_j \alpha_j\delta_{ij} = \alpha_i.$$

In the second place, every y in $\mathcal{V}'$ is a linear combination of $y_1, \cdots, y_n$. To prove this, write $[x_i, y] = \alpha_i$; then, for $x = \sum_i \xi_i x_i$, we have

$$[x, y] = \xi_1\alpha_1 + \cdots + \xi_n\alpha_n.$$

On the other hand

$$[x, y_j] = \sum_i \xi_i[x_i, y_j] = \xi_j,$$

so that, substituting in the preceding equation, we get

$$[x, y] = \alpha_1[x, y_1] + \cdots + \alpha_n[x, y_n]$$

$$= [x, \alpha_1 y_1 + \cdots + \alpha_n y_n].$$

Consequently $y = \alpha_1 y_1 + \cdots + \alpha_n y_n$, and the proof of the theorem is complete.

We shall need also the following easy consequence of Theorem 2.

THEOREM 3. *If u and v are any two different vectors of the n-dimensional vector space $\mathcal{V}$, then there exists a linear functional y on $\mathcal{V}$ such that $[u, y] \neq [v, y]$; or, equivalently, to any non-zero vector x in $\mathcal{V}$ there corresponds a y in $\mathcal{V}'$ such that $[x, y] \neq 0$.*

PROOF. That the two statements in the theorem are indeed equivalent is seen by considering $x = u - v$. We shall, accordingly, prove the latter statement only.

Let $\mathfrak{X} = \{x_1, \cdots, x_n\}$ be any basis in $\mathcal{V}$, and let $\mathfrak{X}' = \{y_1, \cdots, y_n\}$ be the dual basis in $\mathcal{V}'$. If $x = \sum_i \xi_i x_i$, then (as above) $[x, y_j] = \xi_j$. Hence if $[x, y] = 0$ for all y, and, in particular, if $[x, y_j] = 0$ for $j = 1, \cdots, n$, then $x = 0$.

§ 16. Reflexivity

It is natural to think that if the dual space $\mathcal{V}'$ of a vector space $\mathcal{V}$, and the relations between a space and its dual, are of any interest at all for $\mathcal{V}$, then they are of just as much interest for $\mathcal{V}'$. In other words, we propose now to form the dual space $(\mathcal{V}')'$ of $\mathcal{V}'$; for simplicity of notation we shall denote it by $\mathcal{V}''$. The verbal description of an element of $\mathcal{V}''$ is clumsy: such an element is a linear functional of linear functionals. It is, however, at this point that the greatest advantage of the notation $[x, y]$ appears; by means of it, it is easy to discuss $\mathcal{V}$ and its relation to $\mathcal{V}''$.

If we consider the symbol $[x, y]$ for some fixed $y = y_0$, we obtain nothing new: $[x, y_0]$ is merely another way of writing the value $y_0(x)$ of the function y_0 at the vector x. If, however, we consider the symbol $[x, y]$ for some fixed $x = x_0$, then we observe that the function of the vectors in $\mathcal{V}'$, whose value at y is $[x_0, y]$, is a scalar-valued function that happens to be linear (see § 14, (2)); in other words, $[x_0, y]$ defines a linear functional on $\mathcal{V}'$, and, consequently, an element of $\mathcal{V}''$.

By this method we have exhibited *some* linear functionals on $\mathcal{V}'$; have we exhibited them all? For the finite-dimensional case the following theorem furnishes the affirmative answer.

THEOREM. *If $\mathcal{V}$ is a finite-dimensional vector space, then corresponding to every linear functional z_0 on $\mathcal{V}'$ there is a vector x_0 in $\mathcal{V}$ such that $z_0(y) = [x_0, y] = y(x_0)$ for every y in $\mathcal{V}'$; the correspondence $z_0 \rightleftarrows x_0$ between $\mathcal{V}''$ and $\mathcal{V}$ is an isomorphism.*

The correspondence described in this statement is called the *natural correspondence* between $\mathcal{V}''$ and $\mathcal{V}$.

PROOF. Let us view the correspondence from the standpoint of going from $\mathcal{V}$ to $\mathcal{V}''$; in other words, to every x_0 in $\mathcal{V}$ we make correspond a vector z_0 in $\mathcal{V}''$ defined by $z_0(y) = y(x_0)$ for every y in $\mathcal{V}'$. Since $[x, y]$ depends linearly on x, the transformation $x_0 \to z_0$ is linear.

We shall show that this transformation is one-to-one, as far as it goes. We assert, in other words, that if x_1 and x_2 are in $\mathcal{V}$, and if z_1 and z_2 are the corresponding vectors in $\mathcal{V}''$ (so that $z_1(y) = [x_1, y]$ and $z_2(y) = [x_2, y]$ for all y in $\mathcal{V}'$), and if $z_1 = z_2$, then $x_1 = x_2$. To say that $z_1 = z_2$ means that $[x_1, y] = [x_2, y]$ for every y in $\mathcal{V}'$; the desired conclusion follows from § 15, Theorem 3.

The last two paragraphs together show that the set of those linear functionals z on $\mathcal{V}'$ (that is, elements of $\mathcal{V}''$) that do have the desired form (that is, $z(y)$ is identically equal to $[x, y]$ for a suitable x in $\mathcal{V}$) is a subspace of $\mathcal{V}''$ which is isomorphic to $\mathcal{V}$ and which is, therefore, n-dimensional. But the n-dimensionality of $\mathcal{V}$ implies that of $\mathcal{V}'$, which in turn implies that $\mathcal{V}''$ is n-dimensional. It follows that $\mathcal{V}''$ must coincide with the n-dimensional subspace just described, and the proof of the theorem is complete.

It is important to observe that the theorem shows not only that $\mathcal{V}$ and $\mathcal{V}''$ are isomorphic—this much is trivial from the fact that they have the same dimension—but that the natural correspondence is an isomorphism. This property of vector spaces is called *reflexivity;* every finite-dimensional vector space is reflexive.

It is frequently convenient to be mildly sloppy about $\mathcal{V}''$: for finite-dimensional vector spaces we shall identify $\mathcal{V}''$ with $\mathcal{V}$ (by the natural isomorphism), and we shall say that the element z_0 of $\mathcal{V}''$ is the *same* as the element x_0 of $\mathcal{V}$ whenever $z_0(y) = [x_0, y]$ for all y in $\mathcal{V}'$. In this language it is very easy to express the relation between a basis $\mathfrak{X}$, in $\mathcal{V}$, and the dual basis of its dual basis, in $\mathcal{V}''$; the symmetry of the relation $[x_i, y_j] = \delta_{ij}$ shows that $\mathfrak{X}'' = \mathfrak{X}$.

§ 17. Annihilators

DEFINITION. The *annihilator* $\mathcal{S}^0$ of any subset $\mathcal{S}$ of a vector space $\mathcal{V}$ ($\mathcal{S}$ need not be a subspace) is the set of all vectors y in $\mathcal{V}'$ such that $[x, y]$ is identically zero for all x in $\mathcal{S}$.

Thus $\mathcal{O}^0 = \mathcal{V}'$ and $\mathcal{V}^0 = \mathcal{O}$ ($\subset \mathcal{V}'$). If $\mathcal{V}$ is finite-dimensional and $\mathcal{S}$ contains a non-zero vector, so that $\mathcal{S} \neq \mathcal{O}$, then § 15, Theorem 3 shows that $\mathcal{S}^0 \neq \mathcal{V}'$.

THEOREM 1. *If $\mathfrak{M}$ is an m-dimensional subspace of an n-dimensional vector space $\mathcal{V}$, then $\mathfrak{M}^0$ is an $(n - m)$-dimensional subspace of $\mathcal{V}'$.*

PROOF. We leave it to the reader to verify that $\mathfrak{M}^0$ (in fact $\mathcal{S}^0$, for an arbitrary $\mathcal{S}$) is always a subspace; we shall prove only the statement concerning the dimension of $\mathfrak{M}^0$.

Let $\mathcal{X} = \{x_1, \cdots, x_n\}$ be a basis in $\mathcal{V}$ whose first m elements are in $\mathfrak{M}$ (and form therefore a basis for $\mathfrak{M}$); let $\mathcal{X}' = \{y_1, \cdots, y_n\}$ be the dual basis in $\mathcal{V}'$. We denote by $\mathfrak{N}$ the subspace (in $\mathcal{V}'$) spanned by $y_{m+1}, \cdots, y_n$; clearly $\mathfrak{N}$ has dimension $n - m$. We shall prove that $\mathfrak{M}^0 = \mathfrak{N}$.

If x is any vector in $\mathfrak{M}$, then x is a linear combination of $x_1, \cdots, x_m$,

$$x = \sum_{i=1}^{m} \xi_i x_i,$$

and, for any $j = m + 1, \cdots, n$, we have

$$[x, y_j] = \sum_{i=1}^{m} \xi_i [x_i, y_j] = 0.$$

In other words, y_j is in $\mathfrak{M}^0$ for $j = m + 1, \cdots, n$; it follows that $\mathfrak{N}$ is contained in $\mathfrak{M}^0$,

$$\mathfrak{N} \subset \mathfrak{M}^0.$$

Suppose, on the other hand, that y is any element of $\mathfrak{M}^0$. Since y, being in $\mathcal{V}'$, is a linear combination of the basis vectors $y_1, \cdots, y_n$, we may write

$$y = \sum_{j=1}^{n} \eta_j y_j.$$

Since, by assumption, y is in $\mathfrak{M}^0$, we have, for every $i = 1, \cdots, m$,

$$0 = [x_i, y] = \sum_{j=1}^{n} \eta_j [x_i, y_j] = \eta_i;$$

in other words, y is a linear combination of $y_{m+1}, \cdots, y_n$. This proves that y is in $\mathfrak{N}$, and consequently that

$$\mathfrak{M}^0 \subset \mathfrak{N},$$

and the theorem follows.

THEOREM 2. *If $\mathfrak{M}$ is a subspace in a finite-dimensional vector space $\mathcal{V}$, then* $\mathfrak{M}^{00}$ $(=(\mathfrak{M}^0)^0) = \mathfrak{M}$.

PROOF. Observe that we use here the convention, established at the end of § 16, that identifies $\mathcal{V}$ and $\mathcal{V}''$. By definition, $\mathfrak{M}^{00}$ is the set of all vectors x such that $[x, y] = 0$ for all y in $\mathfrak{M}^0$. Since, by the definition of $\mathfrak{M}^0$, $[x, y] = 0$ for all x in $\mathfrak{M}$ and all y in $\mathfrak{M}^0$, it follows that $\mathfrak{M} \subset \mathfrak{M}^{00}$. The desired conclusion now follows from a dimension argument. Let $\mathfrak{M}$ be m-dimensional; then the dimension of $\mathfrak{M}^0$ is $n - m$, and that of $\mathfrak{M}^{00}$ is $n - (n - m) = m$. Hence $\mathfrak{M} = \mathfrak{M}^{00}$, as was to be proved.

EXERCISES

1. Define a non-zero linear functional y on $\mathcal{C}^3$ such that if $x_1 = (1, 1, 1)$ and $x_2 = (1, 1, -1)$, then $[x_1, y] = [x_2, y] = 0$.

2. The vectors $x_1 = (1, 1, 1)$, $x_2 = (1, 1, -1)$, and $x_3 = (1, -1, -1)$ form a basis of $\mathcal{C}^3$. If $\{y_1, y_2, y_3\}$ is the dual basis, and if $x = (0, 1, 0)$, find $[x, y_1]$, $[x, y_2]$, and $[x, y_3]$.

3. Prove that if y is a linear functional on an n-dimensional vector space $\mathcal{V}$, then the set of all those vectors x for which $[x, y] = 0$ is a subspace of $\mathcal{V}$; what is the dimension of that subspace?

4. If $y(x) = \xi_1 + \xi_2 + \xi_3$ whenever $x = (\xi_1, \xi_2, \xi_3)$ is a vector in $\mathcal{C}^3$, then y is a linear functional on $\mathcal{C}^3$; find a basis of the subspace consisting of all those vectors x for which $[x, y] = 0$.

5. Prove that if $m < n$, and if $y_1, \cdots, y_m$ are linear functionals on an n-dimensional vector space $\mathcal{V}$, then there exists a non-zero vector x in $\mathcal{V}$ such that $[x, y_j] = 0$ for $j = 1, \cdots, m$. What does this result say about the solutions of linear equations?

6. Suppose that $m < n$ and that $y_1, \cdots, y_m$ are linear functionals on an n-dimensional vector space $\mathcal{V}$. Under what conditions on the scalars $\alpha_1, \cdots, \alpha_m$ is it true that there exists a vector x in $\mathcal{V}$ such that $[x, y_j] = \alpha_j$ for $j = 1, \cdots, m$? What does this result say about the solutions of linear equations?

7. If $\mathcal{V}$ is an n-dimensional vector space over a finite field, and if $0 \leq m \leq n$ then the number of m-dimensional subspaces of $\mathcal{V}$ is the same as the number of $(n - m)$-dimensional subspaces.

8. (a) Prove that if $\mathcal{S}$ is any subset of a finite-dimensional vector space, then $\mathcal{S}^{00}$ coincides with the subspace spanned by $\mathcal{S}$.
(b) If $\mathcal{S}$ and $\mathfrak{I}$ are subsets of a vector space, and if $\mathcal{S} \subset \mathfrak{I}$, then $\mathfrak{I}^0 \subset \mathcal{S}^0$.
(c) If $\mathfrak{M}$ and $\mathfrak{N}$ are subspaces of a finite-dimensional vector space, then $(\mathfrak{M} \cap \mathfrak{N})^0 = \mathfrak{M}^0 + \mathfrak{N}^0$ and $(\mathfrak{M} + \mathfrak{N})^0 = \mathfrak{M}^0 \cap \mathfrak{N}^0$. (Hint: make repeated use of (b) and of § 17, Theorem 2.)
(d) Is the conclusion of (c) valid for not necessarily finite-dimensional vector spaces?

9. This exercise is concerned with vector spaces that need not be finite-dimensional; most of its parts (but not all) depend on the sort of transfinite reasoning that is needed to prove that every vector space has a basis (cf. § 7, Ex. 11).

(a) Suppose that f and g are scalar-valued functions defined on a set $\mathfrak{X}$; if α and β are scalars write $h = \alpha f + \beta g$ for the function defined by $h(x) = \alpha f(x) + \beta g(x)$ for all x in $\mathfrak{X}$. The set of all such functions is a vector space with respect to this definition of the linear operations, and the same is true of the set of all finitely non-zero functions. (A function f on $\mathfrak{X}$ is *finitely non-zero* if the set of those elements x of $\mathfrak{X}$ for which $f(x) \neq 0$ is finite.)

(b) Every vector space is isomorphic to the set of all finitely non-zero functions on some set.

(c) If $\mathfrak{V}$ is a vector space with basis $\mathfrak{X}$, and if f is a scalar-valued function defined on the set $\mathfrak{X}$, then there exists a unique linear functional y on $\mathfrak{V}$ such that $[x, y] = f(x)$ for all x in $\mathfrak{X}$.

(d) Use (a), (b), and (c) to conclude that every vector space $\mathfrak{V}$ is isomorphic to a subspace of $\mathfrak{V}'$.

(e) Which vector spaces are isomorphic to their own duals?

(f) If $\mathfrak{Y}$ is a linearly independent subset of a vector space $\mathfrak{V}$, then there exists a basis of $\mathfrak{V}$ containing $\mathfrak{Y}$. (Compare this result with the theorem of § 7.)

(g) If $\mathfrak{X}$ is a set and if y is an element of $\mathfrak{X}$, write f_y for the scalar-valued function defined on $\mathfrak{X}$ by writing $f_y(x) = 1$ or 0 according as $x = y$ or $x \neq y$. Let $\mathfrak{Y}$ be the set of all functions f_y together with the function g defined by $g(x) = 1$ for all x in $\mathfrak{X}$. Prove that if $\mathfrak{X}$ is infinite, then $\mathfrak{Y}$ is a linearly independent subset of the vector space of all scalar-valued functions on $\mathfrak{X}$.

(h) The natural correspondence from $\mathfrak{V}$ to $\mathfrak{V}''$ is defined for all vector spaces (not only for the finite-dimensional ones); if x_0 is in $\mathfrak{V}$, define the corresponding element z_0 of $\mathfrak{V}''$ by writing $z_0(y) = [x_0, y]$ for all y in $\mathfrak{V}'$. Prove that if $\mathfrak{V}$ is reflexive (i.e., if every z_0 in $\mathfrak{V}''$ can be obtained in this manner by a suitable choice of x_0), then $\mathfrak{V}$ is finite-dimensional. (Hint: represent $\mathfrak{V}'$ as the set of all scalar-valued functions on some set, and then use (g), (f), and (c) to construct an element of $\mathfrak{V}''$ that is not induced by an element of $\mathfrak{V}$.)

Warning: the assertion that a vector space is reflexive if and only if it is finite-dimensional would shock most of the experts in the subject. The reason is that the customary and fruitful generalization of the concept of reflexivity to infinite-dimensional spaces is not the simple-minded one given in (h).

§ 18. Direct sums

We shall study several important general methods of making new vector spaces out of old ones; in this section we begin by studying the easiest one.

DEFINITION. If $\mathfrak{U}$ and $\mathfrak{V}$ are vector spaces (over the same field), their *direct sum* is the vector space $\mathfrak{W}$ (denoted by $\mathfrak{U} \oplus \mathfrak{V}$) whose elements are all the ordered pairs $\langle x, y \rangle$ with x in $\mathfrak{U}$ and y in $\mathfrak{V}$, with the linear operations defined by

$$\alpha_1\langle x_1, y_1\rangle + \alpha_2\langle x_2, y_2\rangle = \langle \alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2\rangle.$$

We observe that the formation of the direct sum is analogous to the way in which the plane is constructed from its two coordinate axes.

We proceed to investigate the relation of this notion to some of our earlier ones.

The set of all vectors (in $\mathcal{W}$) of the form $\langle x, 0\rangle$ is a subspace of $\mathcal{W}$; the correspondence $\langle x, 0\rangle \rightleftarrows x$ shows that this subspace is isomorphic to $\mathcal{U}$. It is convenient, once more, to indulge in a logical inaccuracy and, identifying x and $\langle x, 0\rangle$, to speak of $\mathcal{U}$ as a subspace of $\mathcal{W}$. Similarly, of course, the vectors y of $\mathcal{V}$ may be identified with the vectors of the form $\langle 0, y\rangle$ in $\mathcal{W}$, and we may consider $\mathcal{V}$ as a subspace of $\mathcal{W}$. This terminology is, to be sure, not quite exact, but the logical difficulty is much easier to get around here than it was in the case of the second dual space. We could have defined the direct sum of $\mathcal{U}$ and $\mathcal{V}$ (at least in the case in which $\mathcal{U}$ and $\mathcal{V}$ have no non-zero vectors in common) as the set consisting of all x's in $\mathcal{U}$, all y's in $\mathcal{V}$, and all those pairs $\langle x, y\rangle$ for which $x \neq 0$ and $y \neq 0$. This definition yields a theory analogous in every detail to the one we shall develop, but it makes it a nuisance to prove theorems because of the case distinctions it necessitates. It is clear, however, that from the point of view of this definition $\mathcal{U}$ is actually a subset of $\mathcal{U} \oplus \mathcal{V}$. In this sense then, or in the isomorphism sense of the definition we did adopt, we raise the question: what is the relation between $\mathcal{U}$ and $\mathcal{V}$ when we consider these spaces as subspaces of the big space $\mathcal{W}$?

THEOREM. *If $\mathcal{U}$ and $\mathcal{V}$ are subspaces of a vector space $\mathcal{W}$, then the following three conditions are equivalent.*

(1) $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$.

(2) $\mathcal{U} \cap \mathcal{V} = \mathcal{O}$ *and* $\mathcal{U} + \mathcal{V} = \mathcal{W}$ (*i.e.,* $\mathcal{U}$ *and* $\mathcal{V}$ *are complements of each other*).

(3) *Every vector z in $\mathcal{W}$ may be written in the form $z = x + y$, with x in $\mathcal{U}$ and y in $\mathcal{V}$, in one and only one way.*

PROOF. We shall prove the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). We assume that $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$. If $z = \langle x, y\rangle$ lies in both $\mathcal{U}$ and $\mathcal{V}$, then $x = y = 0$, so that $z = 0$; this proves that $\mathcal{U} \cap \mathcal{V} = \mathcal{O}$. Since the representation $z = \langle x, 0\rangle + \langle 0, y\rangle$ is valid for every z, it follows also that $\mathcal{U} + \mathcal{V} = \mathcal{W}$.

(2) $\Rightarrow$ (3). If we assume (2), so that, in particular, $\mathcal{U} + \mathcal{V} = \mathcal{W}$, then it is clear that every z in $\mathcal{W}$ has the desired representation, $z = x + y$. To prove uniqueness, we assume that $z = x_1 + y_1$ and $z = x_2 + y_2$, with x_1 and x_2 in $\mathcal{U}$ and y_1 and y_2 in $\mathcal{V}$. Since $x_1 + y_1 = x_2 + y_2$, it follows that $x_1 - x_2 = y_2 - y_1$. Since the left member of this last equation is in $\mathcal{U}$ and the right member is in $\mathcal{V}$, the disjointness of $\mathcal{U}$ and $\mathcal{V}$ implies that $x_1 = x_2$ and $y_1 = y_2$.

(3) $\Rightarrow$ (1). This implication is practically indistinguishable from the definition of direct sum. If we form the direct sum $\mathcal{U} \oplus \mathcal{V}$, and then

identify $\langle x, 0\rangle$ and $\langle 0, y\rangle$ with x and y respectively, we are committed to identifying the sum $\langle x, y\rangle = \langle x, 0\rangle + \langle 0, y\rangle$ with what we are assuming to be the general element $z = x + y$ of $\mathcal{W}$; from the hypothesis that the representation of z in the form $x + y$ is unique we conclude that the correspondence between $\langle x, 0\rangle$ and x (and also between $\langle 0, y\rangle$ and y) is one-to-one.

If two subspaces $\mathcal{U}$ and $\mathcal{V}$ in a vector space $\mathcal{W}$ are disjoint and span $\mathcal{W}$ (that is, if they satisfy (2)), it is usual to say that $\mathcal{W}$ is the *internal direct sum* of $\mathcal{U}$ and $\mathcal{V}$; symbolically, as before, $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$. If we want to emphasize the distinction between this concept and the one defined before, we describe the earlier one by saying that $\mathcal{W}$ is the *external direct sum* of $\mathcal{U}$ and $\mathcal{V}$. In view of the natural isomorphisms discussed above, and, especially, in view of the preceding theorem, the distinction is more pedantic than conceptual. In accordance with our identification convention, we shall usually ignore it.

§ 19. Dimension of a direct sum

What can be said about the dimension of a direct sum? If $\mathcal{U}$ is n-dimensional, $\mathcal{V}$ is m-dimensional, and $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$, what is the dimension of $\mathcal{W}$? This question is easy to answer.

THEOREM 1. *The dimension of a direct sum is the sum of the dimensions of its summands.*

PROOF. We assert that if $\{x_1, \cdots, x_n\}$ is a basis in $\mathcal{U}$, and if $\{y_1, \cdots, y_m\}$ is a basis in $\mathcal{V}$, then the set $\{x_1, \cdots, x_n, y_1, \cdots, y_m\}$ (or, more precisely, the set $\{\langle x_1, 0\rangle, \cdots, \langle x_n, 0\rangle, \langle 0, y_1\rangle, \cdots, \langle 0, y_m\rangle\}$) is a basis in $\mathcal{W}$. The easiest proof of this assertion is to use the implication (1) $\Rightarrow$ (3) from the theorem of the preceding section. Since every z in $\mathcal{W}$ may be written in the form $z = x + y$, where x is a linear combination of $x_1, \cdots, x_n$ and y is a linear combination of $y_1, \cdots, y_m$, it follows that our set does indeed span $\mathcal{W}$. To show that the set is also linearly independent, suppose that

$$\alpha_1 x_1 + \cdots + \alpha_n x_n + \beta_1 y_1 + \cdots + \beta_m y_m = 0.$$

The uniqueness of the representation of 0 in the form $x + y$ implies that

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = \beta_1 y_1 + \cdots + \beta_m y_m = 0,$$

and hence the linear independence of the x's and of the y's implies that

$$\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_m = 0.$$

THEOREM 2. *If $\mathcal{W}$ is any $(n + m)$-dimensional vector space, and if $\mathcal{U}$ is any n-dimensional subspace of $\mathcal{W}$, then there exists an m-dimensional subspace $\mathcal{V}$ in $\mathcal{W}$ such that $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$.*

PROOF. Let $\{x_1, \cdots, x_n\}$ be any basis in $\mathcal{U}$; by the theorem of § 7 we may find a set $\{y_1, \cdots, y_m\}$ of vectors in $\mathcal{W}$ with the property that $\{x_1, \cdots, x_n, y_1, \cdots, y_m\}$ is a basis in $\mathcal{W}$. Let $\mathcal{V}$ be the subspace spanned by $y_1, \cdots, y_m$; we omit the verification that $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$.

Theorem 2 says that every subspace of a finite-dimensional vector space has a complement.

§ 20. Dual of a direct sum

In most of what follows we shall view the notion of direct sum as defined for subspaces of a vector space $\mathcal{V}$; this avoids the fuss with the identification convention of § 18, and it turns out, incidentally, to be the more useful concept for our later work. We conclude, for the present, our study of direct sums, by observing the simple relation connecting dual spaces, annihilators, and direct sums. To emphasize our present view of direct summation, we return to the letters of our earlier notation.

THEOREM. *If $\mathfrak{M}$ and $\mathfrak{N}$ are subspaces of a vector space $\mathcal{V}$, and if $\mathcal{V} = \mathfrak{M} \oplus \mathfrak{N}$, then $\mathfrak{M}'$ is isomorphic to $\mathfrak{N}^0$ and $\mathfrak{N}'$ to $\mathfrak{M}^0$, and $\mathcal{V}' = \mathfrak{M}^0 \oplus \mathfrak{N}^0$.*

PROOF. To simplify the notation we shall use, throughout this proof, x, x', and x^0 for elements of $\mathfrak{M}$, $\mathfrak{M}'$, and $\mathfrak{M}^0$, respectively, and we reserve, similarly, the letters y for $\mathfrak{N}$ and z for $\mathcal{V}$. (This notation is not meant to suggest that there is any particular relation between, say, the vectors x in $\mathfrak{M}$ and the vectors x' in $\mathfrak{M}'$.)

If z' belongs to both $\mathfrak{M}^0$ and $\mathfrak{N}^0$, i.e., if $z'(x) = z'(y) = 0$ for all x and y, then $z'(z) = z'(x + y) = 0$ for all z; this implies that $\mathfrak{M}^0$ and $\mathfrak{N}^0$ are disjoint. If, moreover, z' is any vector in $\mathcal{V}'$, and if $z = x + y$, we write $x^0(z) = z'(y)$ and $y^0(z) = z'(x)$. It is easy to see that the functions x^0 and y^0 thus defined are linear functionals on $\mathcal{V}$ (i.e., elements of $\mathcal{V}'$) belonging to $\mathfrak{M}^0$ and $\mathfrak{N}^0$ respectively; since $z' = x^0 + y^0$, it follows that $\mathcal{V}'$ is indeed the direct sum of $\mathfrak{M}^0$ and $\mathfrak{N}^0$.

To establish the asserted isomorphisms, we make correspond to every x^0 a y' in $\mathfrak{N}'$ defined by $y'(y) = x^0(y)$. We leave to the reader the routine verification that the correspondence $x^0 \to y'$ is linear and one-to-one, and therefore an isomorphism between $\mathfrak{M}^0$ and $\mathfrak{N}'$; the corresponding result for $\mathfrak{N}^0$ and $\mathfrak{M}'$ follows from symmetry by interchanging x and y. (Observe that for finite-dimensional vector spaces the mere existence of an isomorphism between, say, $\mathfrak{M}^0$ and $\mathfrak{N}'$ is trivial from a dimension argument; indeed, the dimensions of both $\mathfrak{M}^0$ and $\mathfrak{N}'$ are equal to the dimension of $\mathfrak{N}$.)

We remark, concerning our entire presentation of the theory of direct sums, that there is nothing magic about the number two; we could have defined the direct sum of any finite number of vector spaces, and we could have proved the obvious analogues of all the theorems of the last three sections, with only the notation becoming more complicated. We serve warning that we shall use this remark later and treat the theorems it implies as if we had proved them.

EXERCISES

1. Suppose that x, y, u, and v are vectors in $\mathcal{C}^4$; let $\mathfrak{M}$ and $\mathfrak{N}$ be the subspaces of $\mathcal{C}^4$ spanned by $\{x, y\}$ and $\{u, v\}$ respectively. In which of the following cases is it true that $\mathcal{C}^4 = \mathfrak{M} \oplus \mathfrak{N}$?

(a) $x = (1, 1, 0, 0)$, $y = (1, 0, 1, 0)$
$u = (0, 1, 0, 1)$, $v = (0, 0, 1, 1)$.
(b) $x = (-1, 1, 1, 0)$, $y = (0, 1, -1, 1)$
$u = (1, 0, 0, 0)$, $v = (0, 0, 0, 1)$.
(c) $x = (1, 0, 0, 1)$, $y = (0, 1, 1, 0)$
$u = (1, 0, 1, 0)$, $v = (0, 1, 0, 1)$.

2. If $\mathfrak{M}$ is the subspace consisting of all those vectors $(\xi_1, \cdots, \xi_n, \xi_{n+1}, \cdots, \xi_{2n})$ in $\mathcal{C}^{2n}$ for which $\xi_1 = \cdots = \xi_n = 0$, and if $\mathfrak{N}$ is the subspace of all those vectors for which $\xi_j = \xi_{n+j}$, $j = 1, \cdots, n$, then $\mathcal{C}^{2n} = \mathfrak{M} \oplus \mathfrak{N}$.

3. Construct three subspaces $\mathfrak{M}$, $\mathfrak{N}_1$, and $\mathfrak{N}_2$ of a vector space $\mathcal{V}$ so that $\mathfrak{M} \oplus \mathfrak{N}_1 = \mathfrak{M} \oplus \mathfrak{N}_2 = \mathcal{V}$ but $\mathfrak{N}_1 \neq \mathfrak{N}_2$. (Note that this means that there is no cancellation law for direct sums.) What is the geometric picture corresponding to this situation?

4. (a) If $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are vector spaces, what is the relation between $\mathcal{U} \oplus (\mathcal{V} \oplus \mathcal{W})$ and $(\mathcal{U} \oplus \mathcal{V}) \oplus \mathcal{W}$ (i.e., in what sense is the formation of direct sums an associative operation)?

(b) In what sense is the formation of direct sums commutative?

5. (a) Three subspaces $\mathfrak{L}$, $\mathfrak{M}$, and $\mathfrak{N}$ of a vector space $\mathcal{V}$ are called *independent* if each one is disjoint from the sum of the other two. Prove that a necessary and sufficient condition for $\mathcal{V} = \mathfrak{L} \oplus (\mathfrak{M} \oplus \mathfrak{N})$ (and also for $\mathcal{V} = (\mathfrak{L} \oplus \mathfrak{M}) \oplus \mathfrak{N}$) is that $\mathfrak{L}$, $\mathfrak{M}$, and $\mathfrak{N}$ be independent and that $\mathcal{V} = \mathfrak{L} + \mathfrak{M} + \mathfrak{N}$. (The subspace $\mathfrak{L} + \mathfrak{M} + \mathfrak{N}$ is the set of all vectors of the form $x + y + z$, with x in $\mathfrak{L}$, y in $\mathfrak{M}$, and z in $\mathfrak{N}$.)

(b) Give an example of three subspaces of a vector space $\mathcal{V}$, such that the sum of all three is $\mathcal{V}$, such that every two of the three are disjoint, but such that the three are not independent.

(c) Suppose that x, y, and z are elements of a vector space and that $\mathfrak{L}$, $\mathfrak{M}$, and $\mathfrak{N}$ are the subspaces spanned by x, y, and z, respectively. Prove that the vectors x, y, and z are linearly independent if and only if the subspaces $\mathfrak{L}$, $\mathfrak{M}$, and $\mathfrak{N}$ are independent.

(d) Prove that three finite-dimensional subspaces are independent if and only if the sum of their dimensions is equal to the dimension of their sum.

(e) Generalize the results (a)–(d) from three subspaces to any finite number.

§ 21. Quotient spaces

We know already that if $\mathfrak{M}$ is a subspace of a vector space $\mathfrak{V}$, then there are, usually, many other *subspaces* $\mathfrak{N}$ in $\mathfrak{V}$ such that $\mathfrak{M} \oplus \mathfrak{N} = \mathfrak{V}$. There is no natural way of choosing one from among the wealth of complements of $\mathfrak{M}$. There is, however, a natural construction that associates with $\mathfrak{M}$ and $\mathfrak{V}$ a new vector space that, for all practical purposes, plays the role of a complement of $\mathfrak{M}$. The theoretical advantage that the construction has over the formation of an arbitrary complement is precisely its "natural" character, i.e., the fact that it does not depend on choosing a basis, or, for that matter, on choosing anything at all.

In order to understand the construction it is a good idea to keep a picture in mind. Suppose, for instance, that $\mathfrak{V} = \mathfrak{R}^2$ (the real coordinate plane) and that $\mathfrak{M}$ consists of all those vectors (ξ_1, ξ_2) for which $\xi_2 = 0$ (the horizontal axis). Each complement of $\mathfrak{M}$ is a line (other than the horizontal axis) through the origin. Observe that each such complement has the property that it intersects every horizontal line in exactly one point. The idea of the construction we shall describe is to make a vector space out of the set of all horizontal lines.

We begin by using $\mathfrak{M}$ to single out certain subsets of $\mathfrak{V}$. (We are back in the general case now.) If x is an arbitrary vector in $\mathfrak{V}$, we write $x + \mathfrak{M}$ for the set of all sums $x + y$ with y in $\mathfrak{M}$; each set of the form $x + \mathfrak{M}$ is called a *coset* of $\mathfrak{M}$. (In the case of the plane-line example above, the cosets are the horizontal lines.) Note that one and the same coset can arise from two different vectors, i.e., that even if $x \neq y$, it is possible that $x + \mathfrak{M} = y + \mathfrak{M}$. It makes good sense, just the same, to speak of a coset, say $\mathfrak{H}$, of $\mathfrak{M}$, *without specifying* which element (or elements) $\mathfrak{H}$ comes from; to say that $\mathfrak{H}$ is a coset (of $\mathfrak{M}$) means simply that there is at least one x such that $\mathfrak{H} = x + \mathfrak{M}$.

If $\mathfrak{H}$ and $\mathfrak{K}$ are cosets (of $\mathfrak{M}$), we write $\mathfrak{H} + \mathfrak{K}$ for the set of all sums $u + v$ with u in $\mathfrak{H}$ and v in $\mathfrak{K}$; we assert that $\mathfrak{H} + \mathfrak{K}$ is also a coset of $\mathfrak{M}$. Indeed, if $\mathfrak{H} = x + \mathfrak{M}$ and $\mathfrak{K} = y + \mathfrak{M}$, then every element of $\mathfrak{H} + \mathfrak{K}$ belongs to the coset $(x + y) + \mathfrak{M}$ (note that $\mathfrak{M} + \mathfrak{M} = \mathfrak{M}$), and, conversely, every element of $(x + y) + \mathfrak{M}$ is in $\mathfrak{H} + \mathfrak{K}$. (If, for instance, z is in $\mathfrak{M}$, then $(x + y) + z = (x + z) + (y + 0)$.) In other words, $\mathfrak{H} + \mathfrak{K} = (x + y) + \mathfrak{M}$, so that $\mathfrak{H} + \mathfrak{K}$ is a coset, as asserted. We leave to the reader the verification that coset addition is commutative and associative. The coset $\mathfrak{M}$ (i.e., $0 + \mathfrak{M}$) is such that $\mathfrak{H} + \mathfrak{M} = \mathfrak{H}$ for every coset $\mathfrak{H}$, and, moreover, $\mathfrak{M}$ is the only coset *with this property*. (If $(x + \mathfrak{M}) + (y + \mathfrak{M}) = x + \mathfrak{M}$, then $x + \mathfrak{M}$ contains $x + y$, so that $x + y = x + u$ for some u in $\mathfrak{M}$; this implies that y is in $\mathfrak{M}$, and hence that $y + \mathfrak{M} = \mathfrak{M}$.) If $\mathfrak{H}$ is a coset, then the set consisting of all the vectors $-u$, with u in $\mathfrak{H}$,